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Discussion of Muskingum method parameter *X*

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Abstract: The parameter X of the Muskingum method is a physical parameter that reflects the flood peak attenuation and hydrograph shape flattening of a diffusion wave in motion. In this paper, the historic process that hydrologists have undergone to find a physical explanation of this parameter is briefly discussed. Based on the fact that the Muskingum method is the second-order accuracy difference solution to the diffusion wave equation, its numerical stability condition is analyzed, and a conclusion is drawn: $X \leqslant 0.5$ is the uniform condition satisfying the demands for its physical meaning and numerical stability. It is also pointed out that the methods that regard the sum of squares of differences between the calculated and observed discharges or stages as the objective function and the routing coefficients C_0, C_1 and C_2 of the Muskingum method as the optimization parameters cannot guarantee the physical meaning of X.

Key words: *Muskingum method parameter X*; *physical meaning*; *numerical analysis*; *stability condition*; *parameter calibration*

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1 Introduction

The Muskingum method is one of the most popular flood routing methods. Previously established on the basis of the channel storage equation under empirical assumptions, and actually having very general applicability, this method is undoubtedly attracting hydrologists' research interest. No less than one thousand research papers about the Muskingum method have been published in academic journals all over the world. What are the physical meanings of the routing formula and parameters of the Muskingum method? How can a parameter be calibrated without losing its physical meaning? What is the stability condition of the numerical calculation of the formula? How can the Muskingum method be properly used under different conditions? Hydrologists have been expected to answer all of these questions for a long time. Some theoretical problems of the Muskingum method and its successive routing have been discussed in a previous paper (Rui 2002). This paper discusses further study that has been carried out on parameter *X* of the Muskingum method.

2 Physical explanation of parameter *X*

The original explanation of the Muskingum method is that, for a certain river reach, there

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is a unique Q' corresponding to channel storage volume W as long as an appropriate parameter X can be acquired. Q' is expressed as $Q' = XI + (1 - X)O$, where I and O are inflow to the upstream section and outflow from the downstream section of the river reach, respectively, X is equivalent to a weight, and Q' is discharge for representative channel storage (Chow 1964). Obviously, this explanation does not reveal the problem's essence.

In 1958, Russian hydrologists Kalinin and Miljukov discovered a physical parameter called characteristic river length when they studied the flood wave movement along a river course (HDECWCC 1962). They proved that, for the characteristic river length, the channel storage volume has a single-valued relation with the outflow from the downstream section, but for a river with a length greater or lower than the characteristic river length, the channel storage volume has a loop-shaped curve relation with the outflow from the downstream section. Furthermore, the former and latter loops go opposite directions. In the Muskingum method, there must be an available parameter X that causes the channel storage volume of a certain reach to have a single-valued relation with the outflow from its downstream section. In addition, it has already been proven that the channel storage volume of the characteristic river length definitely has a single-valued relation with the outflow from its downstream section. This means that parameter X must have some theoretical relation with the characteristic river length. The relation deduced by Kalinin and Miljukov (HDECWCC 1962) is

$$X = \frac{1}{2} - \frac{l}{2L} \tag{1}$$

where l is the characteristic river length, and L is the reach length. Eq. (1) has been proven correct by Zhao (1984) and Rui (2002) in different ways.

Eq. (1) reveals that $L = l$ only when $X = 0$; however, if $L > l$ or $L < l$, only when $0 < X < 0.5$ or when $X < 0$ can Q' show a single-valued relation with W. This indicates that the fundamental assumption of the Muskingum method objectively corresponds with the motion law of a flood wave in some rivers.

In 1969, the French hydraulic scientist Cunge found numerical diffusion phenomena while researching the numerical solution to the kinematic wave equation (Cunge 1969). He pointed out that if numerical diffusion is controlled by physical diffusion, that is, if the numerical diffusion coefficient of a kinematic wave equals the physical diffusion coefficient of a diffusion wave, then the first-order accuracy numerical solution to the kinematic wave equation is just the second-order accuracy numerical solution of the diffusion wave equation. Owing to the solution to the first-order accuracy kinematic wave difference equation based on the four-point off-centre difference format being exactly the same as the routing formula of the Muskingum method, the routing formula of the Muskingum method is, from a hydraulic point of view, just the second-order accuracy numerical solution to the diffusion wave equation as long as the following condition can be satisfied:

$$X = \frac{1}{2} - \frac{D}{C\Delta x} \tag{2}$$

where C is the velocity of the diffusion wave, D is the diffusion coefficient of the diffusion wave, and Δx is the step length, also called reach length, namely L in Eq. (1). Eq. (1) and Eq. (2) imply the following relation between D and l:

$$D = \frac{1}{2} Cl \tag{3}$$

In 1978, Koussis concluded from the routing formula of the Muskingum method that its corresponding differential equation is the diffusion wave equation, using the Taylor expansion (Koussis 1978). Since Koussis's research is little known, it is summarized here:

The reach water balance equation and the reach storage equation of the Muskingum method are expressed, respectively, as

$$\frac{\mathrm{d}W}{\mathrm{d}t} = Q(x,t) - Q(x+\Delta x, t) \tag{4}$$

$$W = K[XQ(x,t) + (1-X)Q(x+\Delta x, t)] \tag{5}$$

where $Q(x,t)$ is the inflow to the upstream section of the reach, $Q(x+\Delta x,t)$ is the outflow from the downstream section of the reach, Δx is the step length, W is the reach storage volume, and K is the travel time of the flood wave in the river reach. The Taylor expansion clarifies it:

$$Q(x+\Delta x, t) = Q(x,t) + \Delta x \frac{\partial Q}{\partial x} + \frac{\Delta x^2}{2!} \frac{\partial^2 Q}{\partial x^2} + \cdots \tag{6}$$

Items including Δx^3 and other higher powers in the right part of Eq. (6) can be ignored when Δx is extremely small. In that case,

$$Q(x,t) - Q(x+\Delta x, t) = -\Delta x \frac{\partial Q}{\partial x} - \frac{\Delta x^2}{2!} \frac{\partial^2 Q}{\partial x^2} \tag{7}$$

Substituting Eq. (7) into Eq. (4) and Eq. (5) leads to the following equations:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = -\Delta x \frac{\partial Q}{\partial x} - \frac{\Delta x^2}{2!} \frac{\partial^2 Q}{\partial x^2} \tag{8}$$

and

$$W = K[Q(x,t) + (1-X)\Delta x \frac{\partial Q}{\partial x} + (1-X)\frac{\Delta x^2}{2!} \frac{\partial^2 Q}{\partial x^2}] \tag{9}$$

Differentiating Eq. (9) with respect to t, then substituting it into Eq. (8) gives us

$$K\frac{\partial Q}{\partial t} + K(1-X)\Delta x \frac{\partial^2 Q}{\partial x \partial t} = -\Delta x \frac{\partial Q}{\partial x} - \frac{\Delta x^2}{2!} \frac{\partial^2 Q}{\partial x^2} \tag{10}$$

From the kinematic wave equation, we have

$$\frac{\partial^2 Q}{\partial x \partial t} = -C \frac{\partial^2 Q}{\partial x^2} \tag{11}$$

where C is the velocity of the kinematic wave, equivalent to the velocity of the diffusion wave.

Substituting Eq. (11) and $C = \frac{\Delta x}{K}$ into Eq. (10) and rearranging give us

$$\frac{\partial Q}{\partial t} + C\frac{\partial Q}{\partial x} - [(1-X)\Delta x C - \frac{C}{2}\Delta x]\frac{\partial^2 Q}{\partial x^2} = 0 \tag{12}$$

If it is assumed that

$$(1-X)\Delta x C-\frac{C}{2}\Delta x=D$$

and
$$X=\frac{1}{2}-\frac{D}{C\Delta x} \tag{13}$$

where D is the diffusion coefficient of the diffusion wave, then Eq. (11) is converted into

$$\frac{\partial Q}{\partial t}+C\frac{\partial Q}{\partial x}=D\frac{\partial^2 Q}{\partial x^2} \tag{14}$$

Eq. (14) is just the diffusion wave equation. This result also indicates that the routing formula of the Muskingum method is the diffusion wave equation in differential form. Compared with Cunge's research, it is different in approach but has the same result. The fact that the routing formula of the Muskingum method completely corresponds with the second-order accuracy numerical solution of the diffusion wave equation if $X=\frac{1}{2}-\frac{D}{C\Delta x}$, which can be proven from both positive and negative aspects, provides a foundation for improving the theoretical explanation of the Muskingum method.

3 Relationship between *X* and numerical stability condition

The idea that the value of X is related to the stability of the numerical solution occurred to the authors when they read a paper by Cunge (1969). The numerical test shown in Figure 1 was executed and it was found that the numerical solution is stable when $X \leqslant 0.5$ and unstable when $X > 0.5$.

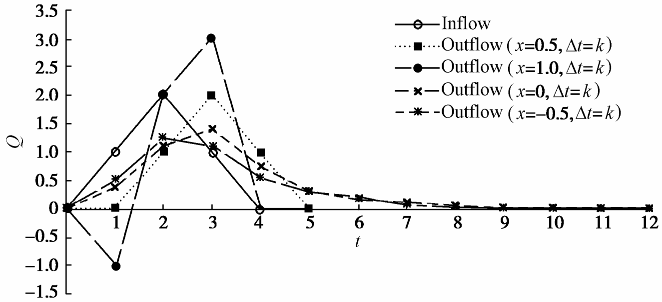

Figure 1 Results of numerical simulation

If the four-point off-centre explicit difference format (Figure 2) is adopted, then

$$\frac{\partial Q}{\partial t}=\frac{X(Q_i^{j+1}-Q_i^j)+(1-X)(Q_{i+1}^{j+1}-Q_{i+1}^j)}{\Delta t} \tag{15}$$

$$\frac{\partial Q}{\partial x}=\frac{(Q_{i+1}^{j+1}-Q_i^{j+1})+(Q_{i+1}^j-Q_i^j)}{2\Delta x} \tag{16}$$

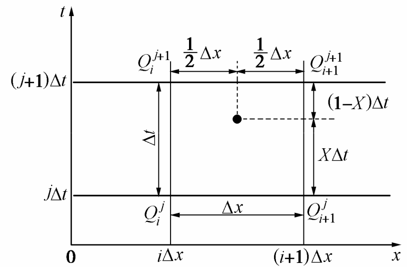

Figure 2 Four-point off-centre difference scheme

The kinematic wave equation can then be discretized as the following difference equation:

$$\frac{X(Q_i^{j+1}-Q_i^j)+(1-X)(Q_{i+1}^{j+1}-Q_{i+1}^j)}{\Delta t}+C\frac{(Q_{i+1}^{j+1}-Q_i^{j+1})+(Q_{i+1}^j-Q_i^j)}{2\Delta x}=0 \tag{17}$$

Q_{i+1}^{j+1} can be solved from Eq. (14):

$$Q_{i+1}^{j+1}=C_0Q_i^j+C_1Q_i^{j+1}+C_2Q_{i+1}^j \tag{18}$$

where

$$\begin{aligned} C_0&=\frac{KX+0.5\Delta t}{K(1-X)+0.5\Delta t}\\ C_1&=\frac{0.5\Delta t-KX}{K(1-X)+0.5\Delta t}\\ C_2&=\frac{K(1-X)-0.5\Delta t}{K(1-X)+0.5\Delta t}\end{aligned} \tag{19}$$

The forms of Eq. (18) and Eq. (19) are exactly the same as the form of the Muskingum flood routing formula. Eq. (18) is deduced from Eq. (17), which is not conditionally stable for its explicit difference format. Therefore, the stability condition for using Eq. (18) and Eq. (19) in flood routing can be derived by analyzing the stability condition of Eq. (17).

Using the Fourier series expansion, Q_i^j can be expressed as

$$Q_i^j=\xi^j\exp(\mathrm{m}ip\Delta x) \tag{20}$$

where $\mathrm{m}=\sqrt{-1}$ is the imaginary unit, p and ξ are the modulus, and the meanings of the other symbols are the same as before. It is easy to prove that

$$Q_i^{j+1}=\xi Q_i^j \tag{21}$$

Substituting Eq. (20) and Eq. (21) into Eq. (17) gives

$$\begin{aligned}&\frac{X}{\Delta t}\left[\xi^{j+1}\exp\left(\mathrm{m}ip\Delta x\right)-\xi^j\exp\left(\mathrm{m}ip\Delta x\right)\right]+\\ &\frac{1-X}{\Delta t}\left\{\xi^{j+1}\exp\left[\mathrm{m}\left(i+1\right)p\Delta x\right]-\xi^j\exp\left[\mathrm{m}\left(i+1\right)p\Delta x\right]\right\}+\\ &\frac{C}{2\Delta x}\left\{\xi^{j+1}\exp\left[\mathrm{m}\left(i+1\right)p\Delta x\right]-\xi^{j+1}\exp\left(\mathrm{m}ip\Delta x\right)\right\}+\\ &\frac{C}{2\Delta x}\left\{\xi^j\exp\left[\mathrm{m}\left(i+1\right)p\Delta x\right]-\xi^j\exp\left(\mathrm{m}ip\Delta x\right)\right\}=0\end{aligned} \tag{22}$$

Eq. (22) can be written as

$$\xi[(X-\frac{r}{2})+\exp(\mathrm{m}p\Delta x)(1-X+\frac{r}{2})]=X+\frac{r}{2}+\exp(\mathrm{m}p\Delta x)(1-X-\frac{r}{2}) \tag{23}$$

where $r=\frac{C\Delta t}{\Delta x}$ is the Courant number, and the meanings of the other symbols are the same as before. Assuming that $a=X-\frac{r}{2}$, $b=1-X+\frac{r}{2}$, $c=\cos(p\Delta x)$, and $d=\sin(p\Delta x)$, Eq. (23) can be written as

$$\xi(a+bc+\mathrm{m}bd)=a+r+(b-r)c+\mathrm{m}(b-r)d \tag{24}$$

and

$$|\xi|^2=\frac{\left[a+r+(b-r)c\right]^2+(b-r)^2d^2}{(a+bc)^2+b^2d^2} \tag{25}$$

Ven Neumamm testified that the condition making Eq. (17) stable is $|\xi|\leqslant 1$ (Wood 1993). Hence, the following result can be deduced from Eq. (24):

$$\left[a+r+(b-r)c\right]^2+(b-r)^2d^2\leqslant(a+bc)^2+b^2d^2 \tag{26}$$

and

$$(a-b+r)(1-c)\leqslant 0 \tag{27}$$

Since $(1-c)=1-\cos(p\Delta x)$ is non-negative, there must be that

$$a-b+r\leqslant 0 \tag{28}$$

Substituting the expressions of *a* and *b* back into Eq. (22) gives

$$X\leqslant 0.5 \tag{29}$$

Eq. (29) shows that only if $X\leqslant 0.5$ can the stability of the numerical calculation be ensured when Eq. (18) and Eq. (19) are used for flood routing. Otherwise, invalid results deviating from the physical meaning will come out. The mentioned measure of making the numerical diffusion coefficient equal to the physical diffusion coefficient aimed at guaranteeing flood peak attenuation and flattening during the diffusion wave's propagation. Therefore, the result $X\leqslant 0.5$ can also be obtained by analyzing the physical meaning. The exactly same conclusion has been drawn from both physical analysis and mathematical analysis.

4 Calibration of parameter *X*

McCarthy proposed the trial-and-error method for calibrating the two parameters *X* and *K* when he established the Muskingum method (Chow 1964). Given various *X* values, the value that gives Q' a single-valued relation with *W* is the required *X*, and the gradient of the corresponding relation curve of Q' and *W* is the value of *K*. The aforementioned is just the basic concept of the trial-and-error method. This way of acquiring the parameters of the Muskingum method is direct as well as being in accordance with the original idea of Muskingum method. However, this calculation method is comparatively complicated and subjective to some extent.

In the 1960s, Zhong (1963) suggested replacing the trial-and-error method with the least-square method in order to avoid the complexity and subjectivity of the calculation. In this least-square method, the objective function is the minimization of the match error for making the relation between Q' and W as single-valued as possible, and the calculation formulas of X and K are obtained according to the condition of the reach water balance:

$$X = -\frac{\sum_{i=1}^{n}[(I_i - O_i)O_i]}{\sum_{i=1}^{n}(I_i - O_i)^2} \tag{30}$$

$$K = -\frac{n\sum_{i=1}^{n}(W_i O_i) - \sum_{i=1}^{n} W_i \sum_{i=1}^{n} O_i}{n\left[\sum_{i=1}^{n} O_i^2 + 2X\sum_{i=1}^{n} O_i(I_i - O_i) + X^2\sum_{i=1}^{n}(I_i - O_i)^2\right] - (\sum_{i=1}^{n} O_i)^2} \tag{31}$$

where I_i, O_i, and W_i are the inflow, outflow, and storage volume of the reach at time $i\Delta t$, respectively, Δt is the time interval, and n is the number of time intervals during a flood. Obviously, using Eq. (30) and Eq. (31) to calibrate the parameters of the Muskingum method can still maintain the essential meanings of X and K.

In recent years, two methods worthy of discussion emerged when the optimization method was used to calibrate the parameters of the Muskingum method. The first regards the routing coefficients C_0, C_1 and C_2 of the Muskingum method as the parameters, the sum of squares of differences between calculated and measured discharges as the objective function, and $C_0 + C_1 + C_2 = 1$ as the constraint condition of reach water balance, and ascertains C_0, C_1 and C_2 with the optimization method. The second regards C_0, C_1 and C_2 as the parameters and $C_0 + C_1 + C_2 = 1$ as the constraint condition, but regards the sum of squares of differences between calculated and measured stages as the objective function. As for the parameters X and K of the Muskingum method, both methods have some obvious flaws. The methods that regard C_0, C_1 and C_2 as the parameters for optimal selection have difficulty maintaining the physical meanings of X and K. That is to say that the physical meanings of X and K are lost in the process although they have already been explained clearly. The method that regards the optimization of the stage process fitting as the objective function is bound to induce an error of conversion between stage and discharge, and will make the physical meanings of X and K much more indistinct. Consequently, it is improper to call flood routing methods built up in such a way the Muskingum method. These should only be called black-box methods.

5 Conclusions

After much long-term research, hydrologists have proven that the assumption about the channel storage equation of the original Muskingum method corresponds with the physical character of diffusion wave motion. The scientific assumption will be raised to a theoretical height once it is supported by a theoretical foundation. Therefore, the Muskingum method should not be regarded as an empirical method anymore.

The key parameter X of the Muskingum method is a physical parameter that reflects flood peak attenuation and shape flattening of a diffusion wave in motion. Moreover, $X \leqslant 0.5$ is the uniform precondition for representing the physical meaning of the Muskingum method and satisfying the stability demand of its numerical calculation. The calculation formula of X deduced in terms of the physical meaning of the Muskingum method can not only be used conveniently in gauged basins, but also provide a helpful tool for calibration of X in un-gauged basins.

The method, which regards the sum of squares of differences between calculated and measured discharges or stages as the objective function and C_0, C_1 and C_2 as the parameters of the Muskingum method for optimal selection, is incapable of ensuring the physical meaning of X, so it can only be classified as a black-box method.

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