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Stochastic back analysis of permeability coefficient using generalized Bayesian method

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Abstract: Owing to the fact that the conventional deterministic back analysis of the permeability coefficient cannot reflect the uncertainties of parameters, including the hydraulic head at the boundary, the permeability coefficient and measured hydraulic head, a stochastic back analysis taking consideration of uncertainties of parameters was performed using the generalized Bayesian method. Based on the stochastic finite element method (SFEM) for a seepage field, the variable metric algorithm and the generalized Bayesian method, formulas for stochastic back analysis of the permeability coefficient were derived. A case study of seepage analysis of a sluice foundation was performed to illustrate the proposed method. The results indicate that, with the generalized Bayesian method that considers the uncertainties of measured hydraulic head, the permeability coefficient and the hydraulic head at the boundary, both the mean and standard deviation of the permeability coefficient can be obtained and the standard deviation is less than that obtained by the conventional Bayesian method. Therefore, the present method is valid and applicable.

Key words: *permeability coefficient*; *stochastic back analysis*; *generalized Bayesian method*; *variable metric algorithm*

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1 Introduction

For seepage analysis, it is crucial to determine the permeability coefficient, which may directly affect the distribution of hydraulic head, flow velocity, and other variables. The conventional method for determining the permeability coefficient is to apply the J. Dupuit formula or the C. V. formula through in situ water pressure tests. However, these two formulas work only when the seepage medium and boundary conditions are simple. In engineering practice, the seepage medium and boundary conditions are often complex, especially for a heterogeneous and anisotropic fractured rock mass, so it is difficult to obtain the permeability coefficient and permeability tensor with these two formulas. For this reason, back analysis of the permeability coefficient based on small amounts of tests, i.e., ascertaining the permeability coefficient through data that are easily monitored, such as the hydraulic head in a piezometric tube, has become a new research direction. The main numerical methods of deterministic back

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analysis are the pulse spectrum method, numerical optimization method, and artificial neural network method. The pulse spectrum method was introduced in seepage parameter identification by Jin and Chen (1991). The transmissibility coefficient of the aquifer was obtained using the inversion method of Sawyer et al. (1995) and Mukhopadhyay (1999), which is based on the artificial neural network method. In order to overcome the shortcomings of a slow convergence speed and a tendency to fall into the local minimum of classical neural networks, a new alternative and iterative algorithm of neural networks based on simulated annealing and radial basic function neural networks (RBFNN) was proposed by Liu et al. (2004). The genetic algorithm and neural networks were combined by He et al. (2004) for inversion of the permeability coefficient of a rock mass. Of the numerical optimization algorithms, the quasi-linear and nonlinear optimization algorithms are commonly used at present. The improved genetic algorithm was proposed by Liu et al. (2003a) for inversion of the permeability coefficient. The real-coded accelerating genetic algorithm (RAGA) was applied to the inversion of hydrogeological parameters by Huo et al. (2004). The genetic algorithm and simulated annealing algorithm were combined by Yang et al. (2005) to ensure global optimization and improve the convergence speed. The complex method was applied to inversion of the permeability coefficient by Wang et al. (2002).

However, the output of back analysis is usually uncertain because of the random factors existing in the problem. Thus, deterministic back analysis is inefficient and stochastic back analysis that accounts for random factors has therefore become a research focus. Geostatistical inversion has been proposed to analyze the reservoir characterization process (Sancevero et al. 2008). The spectral stochastic finite element approach has been utilized for inversion of the Robin coefficient for steady-state heat conduction (Jin and Zou 2008). The generalized Bayesian method has been utilized in stochastic back analysis of mechanical parameters of a rock mass by considering the uncertainty of load and deformation (Huang and Sun 1994). A stochastic back analysis of the thermal parameters of a transient temperature field of a mass of concrete has been performed using Bayesian theory (Liu et al. 2003b). The location and strength of a contaminant source has also been recovered by Bayesian inversion (Yee et al. 2008). However, there has been little research on stochastic back analysis of a seepage field although the back analysis of hydraulic parameters using the stochastic perturbation method has been studied (Yao and Ning 2007). In this study, stochastic back analysis of the permeability coefficient in a two-dimensional steady confined seepage field using the generalized Bayesian method combined with the variable metric method was performed and the uncertainties of the hydraulic head at the boundary, permeability coefficient and measured hydraulic head were considered.

2 Finite element method for seepage field

2.1 Solution of hydraulic head *H* of seepage field

For steady seepage in a horizontal plane, we assume that the rainfall infiltration or evaporation denoted as ω is constant, and that the x and y directions are the main infiltration directions. If the boundary flux is known and the isoparametric element is adopted, the governing equation of an element can be established with the Galerkin method:

$$\sum \boldsymbol{K}^{\mathrm{e}} \boldsymbol{h}^{\mathrm{e}} = \sum \boldsymbol{f}^{\mathrm{e}} \tag{1}$$

where $\boldsymbol{K}^{\mathrm{e}}$ is the element conductivity matrix, $\boldsymbol{h}^{\mathrm{e}}$ is the array of hydraulic head of the element, and $\boldsymbol{f}^{\mathrm{e}}$ is the array of equivalent flux at nodes of the element. Their components are

$$K_{ij}^{\mathrm{e}} = \iint \boldsymbol{B}_i^{\mathrm{T}} \boldsymbol{k} \boldsymbol{B}_j \mathrm{d}\Omega$$

$$f_i^{\mathrm{e}} = -\iint \omega N_i \mathrm{d}\Omega - \int q N_i \mathrm{d}s$$

$$\boldsymbol{B}_i^{\mathrm{T}} = \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_i}{\partial y} \end{bmatrix}, \quad \boldsymbol{k} = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix}$$

where i and j are 1, 2, 3, or 4, N_i is the shape function, Ω is the infiltration domain of the element, s is the boundary of the infiltration domain, q is the flux at the boundary, T is the symbol of transpose, $\boldsymbol{k}$ is the permeability tensor, and k_x and k_y are the permeability coefficients in the x and y directions.

Assembling all governing equations of elements, the global governing equation can be established as

$$\boldsymbol{KH} = \boldsymbol{F} \tag{2}$$

The distribution of the hydraulic head of the seepage field, expressed as an array $\boldsymbol{H}$, can be obtained with Eq. (2), in which $\boldsymbol{K}$ is the global conductivity matrix and $\boldsymbol{F}$ is the global array of equivalent flux at nodes.

2.2 Solution of partial derivative of hydraulic head with respect to permeability coefficient

Here, $\dfrac{\partial \boldsymbol{H}}{\partial \boldsymbol{k}}$, utilized in stochastic back analysis of the permeability coefficient, is derived. After partial derivation with respect to the permeability coefficients k_x and k_y on both sides of Eq. (2), we can derive

$$\frac{\partial \boldsymbol{K}}{\partial k_x}\boldsymbol{H} + \boldsymbol{K}\frac{\partial \boldsymbol{H}}{\partial k_x} = \frac{\partial \boldsymbol{F}}{\partial k_x}, \qquad \frac{\partial \boldsymbol{K}}{\partial k_y}\boldsymbol{H} + \boldsymbol{K}\frac{\partial \boldsymbol{H}}{\partial k_y} = \frac{\partial \boldsymbol{F}}{\partial k_y} \tag{3}$$

Then, we have

$$\frac{\partial \boldsymbol{H}}{\partial k_x} = \boldsymbol{K}^{-1}\left(\frac{\partial \boldsymbol{F}}{\partial k_x} - \frac{\partial \boldsymbol{K}}{\partial k_x}\boldsymbol{H}\right), \qquad \frac{\partial \boldsymbol{H}}{\partial k_y} = \boldsymbol{K}^{-1}\left(\frac{\partial \boldsymbol{F}}{\partial k_y} - \frac{\partial \boldsymbol{K}}{\partial k_y}\boldsymbol{H}\right) \tag{4}$$

In the finite element method, the global conductivity matrix is assembled through each element's conductivity matrix:

$$\boldsymbol{K} = \sum \boldsymbol{K}^{\mathrm{e}} \tag{5}$$

Therefore,

$$\frac{\partial \boldsymbol{K}}{\partial k_x} = \sum \frac{\partial \boldsymbol{K}^{\mathrm{e}}}{\partial k_x}, \qquad \frac{\partial \boldsymbol{K}}{\partial k_y} = \sum \frac{\partial \boldsymbol{K}^{\mathrm{e}}}{\partial k_y} \tag{6}$$

The element conductivity matrix is

$$\boldsymbol{K}^{\mathrm{e}} = \iint \boldsymbol{B}^{\mathrm{T}} \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} \boldsymbol{B} \mathrm{d}\Omega \tag{7}$$

The partial derivative of the element conductivity matrix with respect to the permeability coefficient can be derived:

$$\frac{\partial \boldsymbol{K}^{\mathrm{e}}}{\partial k_x} = \iint \boldsymbol{B}^{\mathrm{T}} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{B} \mathrm{d}\Omega, \qquad \frac{\partial \boldsymbol{K}^{\mathrm{e}}}{\partial k_y} = \iint \boldsymbol{B}^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{B} \mathrm{d}\Omega \tag{8}$$

The partial derivative of hydraulic head with respect to the permeability coefficient can be derived from Eqs. (4) and (8).

3 Stochastic back analysis of permeability coefficient

In stochastic back analysis, the permeability coefficient and measured hydraulic head are no longer treated as deterministic variables but as random variables, so both the mean and standard deviation of the permeability coefficient can be derived (Cividini et al. 1983). The stochastic inversion method can be classified as the Gauss-Markov method, conventional Bayesian method and generalized Bayesian method. Only the uncertainty of the measured hydraulic head is taken into account in the Gauss-Markov method. The conventional Bayesian method considers the uncertainties of both the measured hydraulic head and the permeability coefficient. The generalized Bayesian method considers the uncertainties of these two variables as well as the uncertainty of the hydraulic head at the boundary. Therefore, the generalized Bayesian method combined with the variable metric algorithm in this study was most reasonable for the stochastic back analysis.

3.1 Optimization principle of variable metric algorithm

The variable metric algorithm is an optimization algorithm, in which the new search direction is generated by continuously changing the spatial scale (matrix) during the optimization process to make the initial point converge to an optimal point (Fletcher and Powell 1963). The iterative process is as follows:

(1) An initial point $\boldsymbol{p}_i$ and an $n \times n$ positive definite matrix $\boldsymbol{A}_i$ (commonly a unit matrix) are assumed, and the initial iteration number is set at i =1.

(2) The gradient of error function ∇J_i is computed at the point $\boldsymbol{p}_i$ and search direction $\boldsymbol{S}_i$ is set as

$$\boldsymbol{S}_i = -\boldsymbol{A}_i \nabla J_i \tag{9}$$

(3) The optimal step λ_i is searched for in the direction $\boldsymbol{S}_i$, and $\boldsymbol{p}_{i+1}$ is defined as

$$\boldsymbol{p}_{i+1} = \boldsymbol{p}_i + \lambda_i \boldsymbol{S}_i \tag{10}$$

(4) The optimality of point $\boldsymbol{p}_{i+1}$ is verified. The iterative process is terminated if error

function J_{i+1} is less than a given positive small value ε; otherwise, it proceeds to the next step.

(5) The positive definite matrix is modified as

$$\boldsymbol{A}_{i+1} = \boldsymbol{A}_i + \boldsymbol{M}_i + \boldsymbol{N}_i \tag{11}$$

where

$$\boldsymbol{M}_i = \lambda_i \frac{\boldsymbol{S}_i \boldsymbol{S}_i^{\mathrm{T}}}{\boldsymbol{S}_i^{\mathrm{T}} \boldsymbol{Q}_i}, \qquad \boldsymbol{N}_i = -\frac{(\boldsymbol{A}_i \boldsymbol{Q}_i)(\boldsymbol{A}_i \boldsymbol{Q}_i)^{\mathrm{T}}}{\boldsymbol{Q}_i^{\mathrm{T}} \boldsymbol{A}_i \boldsymbol{Q}_i}, \qquad \boldsymbol{Q}_i = \nabla \boldsymbol{p}_{i+1} - \nabla \boldsymbol{p}_i$$

The iteration number *i* is set to $i = i + 1$ and the process returns to step (2).

The range of steps can first be determined when searching for the optimal step. We can Find λ_{i1} and λ_{i2}, which satisfy the conditions $\left.\frac{\mathrm{d}J_i}{\mathrm{d}\lambda_i}\right|_{\lambda_i=\lambda_{i1}} < 0$ and $\left.\frac{\mathrm{d}J_i}{\mathrm{d}\lambda_i}\right|_{\lambda_i=\lambda_{i2}} > 0$, since

$$\left.\frac{\mathrm{d}J_i}{\mathrm{d}\lambda_i}\right|_{\lambda_i=0} = \boldsymbol{S}_i^{\mathrm{T}} \nabla J_i < 0 \tag{12}$$

The value of λ_{i1} can be set to be zero and λ_{i2} can be determined from an initial step until $\left.\frac{\mathrm{d}J_i}{\mathrm{d}\lambda_i}\right|_{\lambda_i=\lambda_{i2}} > 0$ as the search range is continuously enlarged. Then, the search range of the optimal step can be determined and the optimal step can be derived with the golden section method.

3.2 Bayesian method for parameter inversion

In the Bayesian method for parameter inversion, the measured hydraulic head and permeability coefficient are both considered random variables. It is assumed that $\boldsymbol{H}$ is the calculated value of the hydraulic head of a seepage field, $\boldsymbol{H}^*$ is the measured value of the hydraulic head, $\Delta\boldsymbol{H}$ is the error between the measured value and calculated value, and $\boldsymbol{k}'$ is an array of permeability coefficients with the normal distribution. The prior information of $\boldsymbol{k}'$ has a mean of $\boldsymbol{k}_0'$ and a covariance of $\boldsymbol{C}_{\boldsymbol{k}'}^0$. Then,

$$\Delta\boldsymbol{H} = \boldsymbol{H}^* - \boldsymbol{H}, \quad E(\Delta\boldsymbol{H}) = 0, \quad E(\Delta\boldsymbol{H}\Delta\boldsymbol{H}^{\mathrm{T}}) = \boldsymbol{C}_{\boldsymbol{H}^*} = \boldsymbol{C}_{\Delta\boldsymbol{H}} \tag{13}$$

In Eq. (13), $E(\Delta\boldsymbol{H})$ is the mathematical expectation of $\Delta\boldsymbol{H}$, and $\boldsymbol{C}_{\boldsymbol{H}^*}$ and $\boldsymbol{C}_{\Delta\boldsymbol{H}}$ are the covariances of $\boldsymbol{H}^*$ and $\Delta\boldsymbol{H}$, respectively. The error function in the Bayesian inversion method can be derived:

$$J = (\boldsymbol{H}^* - \boldsymbol{H})^{\mathrm{T}} \boldsymbol{C}_{\boldsymbol{H}^*}^{-1} (\boldsymbol{H}^* - \boldsymbol{H}) + (\boldsymbol{k}' - \boldsymbol{k}_0')^{\mathrm{T}} (\boldsymbol{C}_{\boldsymbol{k}'}^0)^{-1} (\boldsymbol{k}' - \boldsymbol{k}_0') \tag{14}$$

The partial derivative of error function J with respect to the permeability coefficient can be derived from the equation above:

$$\frac{\partial J}{\partial \boldsymbol{k}'} = 2\left(\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{k}'}\right)^{\mathrm{T}} \boldsymbol{C}_{\boldsymbol{H}^*}^{-1} (\boldsymbol{H} - \boldsymbol{H}^*) + 2(\boldsymbol{C}_{\boldsymbol{k}'}^0)^{-1} (\boldsymbol{k}' - \boldsymbol{k}_0') \tag{15}$$

Based on Eqs. (14) and (15), we can derive the formulas to solve the mean and covariance of the permeability coefficient by combining gradient optimization with the iteration method (Gioda and Sakurai 1987).

$\boldsymbol{H}$ is a function of $\boldsymbol{k}'$ and it can be expressed with a Taylor series expansion at mean $\bar{\boldsymbol{k}}'$. Terms more than two orders are truncated. Then,

$$\boldsymbol{H}(\boldsymbol{k}') = \boldsymbol{H}(\bar{\boldsymbol{k}}') + \boldsymbol{S}(\bar{\boldsymbol{k}}')(\boldsymbol{k}' - \bar{\boldsymbol{k}}') \tag{16}$$

where

$$\boldsymbol{S}(\bar{\boldsymbol{k}}') = \left.\frac{\partial \boldsymbol{H}}{\partial \boldsymbol{k}'}\right|_{\boldsymbol{k}'=\bar{\boldsymbol{k}}'}$$

$\boldsymbol{S}$ is a sensitivity matrix. Eq. (16) is substituted into Eq. (15). Then,

$$\frac{\partial J}{\partial \boldsymbol{k}'} = 2\boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{H}^*}^{-1}(\overline{\boldsymbol{H}} + \boldsymbol{S}\boldsymbol{k}' - \boldsymbol{S}\bar{\boldsymbol{k}}' - \boldsymbol{H}^*) + 2(\boldsymbol{C}_{\boldsymbol{k}'}^{0})^{-1}(\boldsymbol{k}' - \boldsymbol{k}_0') \tag{17}$$

where $\overline{\boldsymbol{H}} = \boldsymbol{H}(\bar{\boldsymbol{k}}')$. The equation above is equal to zero when the error function is at a minimum. In this case, $\boldsymbol{k}'$ is equal to its mean value. Then,

$$\left[\boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{H}^*}^{-1}\boldsymbol{S} + (\boldsymbol{C}_{\boldsymbol{k}'}^{0})^{-1}\right]\boldsymbol{k}' = \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{C}_{\boldsymbol{H}^*}^{-1})(\boldsymbol{H}^* - \overline{\boldsymbol{H}} + \boldsymbol{S}\bar{\boldsymbol{k}}') + (\boldsymbol{C}_{\boldsymbol{k}'}^{0})^{-1}\boldsymbol{k}_0' \tag{18}$$

If

$$\boldsymbol{m} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{H}^*}^{-1}\boldsymbol{S} + (\boldsymbol{C}_{\boldsymbol{k}'}^{0})^{-1}$$

$$\boldsymbol{M} = \left[\boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{H}^*}^{-1}\boldsymbol{S} + (\boldsymbol{C}_{\boldsymbol{k}'}^{0})^{-1}\right]^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{H}^*}^{-1}$$

then the estimated value $\hat{\boldsymbol{k}}'$ of $\boldsymbol{k}'$ can be derived:

$$\hat{\boldsymbol{k}}' = \boldsymbol{m}^{-1}(\boldsymbol{C}_{\boldsymbol{k}'}^{0})^{-1}\boldsymbol{k}_0' + \boldsymbol{M}\boldsymbol{H}^* - \boldsymbol{M}(\overline{\boldsymbol{H}} - \boldsymbol{S}\overline{\boldsymbol{k}'}) \tag{19}$$

Note that $(\boldsymbol{C}_{\boldsymbol{k}'}^{0})^{-1} = \boldsymbol{m} - \boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{H}^*}^{-1}\boldsymbol{S}$. The equation above can be written as

$$\hat{\boldsymbol{k}}' = (\boldsymbol{I} - \boldsymbol{M}\boldsymbol{S})\boldsymbol{k}_0' + \boldsymbol{M}\boldsymbol{H}^* - \boldsymbol{M}(\overline{\boldsymbol{H}} - \boldsymbol{S}\overline{\boldsymbol{k}'}) \tag{20}$$

where $\boldsymbol{I}$ is a unit matrix. The last term in the above equation is deterministic. Assuming that the mean of the prior information of the permeability coefficient, $\boldsymbol{k}_0'$, is independent of the measured hydraulic head, the covariance of the permeability coefficients can be derived:

$$\boldsymbol{C}_{\boldsymbol{k}'} = \boldsymbol{C}_{\hat{\boldsymbol{k}}'} = \left(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{S}\right)\boldsymbol{C}_{\boldsymbol{k}'}^{0}\left(\boldsymbol{I} - \boldsymbol{M}\boldsymbol{S}\right)^{\mathrm{T}} + \boldsymbol{M}\boldsymbol{C}_{\boldsymbol{H}^*}^{-1}\boldsymbol{M}^{\mathrm{T}} \tag{21}$$

According to the symmetry and nonsingularity of $\boldsymbol{C}_{\boldsymbol{k}'}^{0}$ and $\boldsymbol{C}_{\boldsymbol{H}^*}$, the equation above can be written as

$$\boldsymbol{C}_{\boldsymbol{k}'} - \boldsymbol{C}_{\boldsymbol{k}'}^{0} - \boldsymbol{A}_0\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{k}'}^{0} = \left[(\boldsymbol{C}_{\boldsymbol{k}'}^{0})^{-1} + \boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{H}^*}^{-1}\boldsymbol{S}\right]^{-1} \tag{22}$$

where $\boldsymbol{A}_0 = \boldsymbol{C}_{\boldsymbol{k}'}^{0}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{C}_{\boldsymbol{H}^*}^{-1}\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{k}'}^{0}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{S}\boldsymbol{C}_{\boldsymbol{k}'}^{0}\boldsymbol{S}^{\mathrm{T}} + \boldsymbol{C}_{\boldsymbol{H}^*})^{-1}$. Eq. (22) indicates that the covariance of the permeability coefficient consists of two parts: the first part corresponds to prior information of the permeability coefficients, and the second part corresponds to the uncertainty of the measured hydraulic head.

3.3 Generalized Bayesian method for parameter inversion

As mentioned above, the generalized Bayesian method is more reasonable than the conventional Bayesian method because the uncertainty of the hydraulic head at the boundary is also considered. Because the uncertainty of the hydraulic head at the boundary of the infiltration domain is considered, the measured value of hydraulic head can be expressed as

$$\boldsymbol{H}^{*}=\boldsymbol{H}(\bar{\boldsymbol{h}}+\Delta\boldsymbol{h}|_{\boldsymbol{k}'})+\Delta\boldsymbol{H}_{0} \tag{23}$$

where $\bar{\boldsymbol{h}}$ is the mean of the hydraulic head at the boundary, $\Delta\boldsymbol{h}$ is the perturbation of the hydraulic head at the boundary, and $\Delta\boldsymbol{H}_0$ is the measurement error of the hydraulic head. Then, the error between the measured value and the calculated value of the hydraulic head of the seepage field $\Delta\boldsymbol{H}$ is

$$\Delta\boldsymbol{H}=\boldsymbol{H}^{*}-\boldsymbol{H}(\bar{\boldsymbol{h}}|_{\boldsymbol{k}'})=\boldsymbol{H}(\bar{\boldsymbol{h}}+\Delta\boldsymbol{h}|_{\boldsymbol{k}'})-\boldsymbol{H}(\bar{\boldsymbol{h}}|_{\boldsymbol{k}'})+\Delta\boldsymbol{H}_{0} \tag{24}$$

If the coefficient of variation of the random variable is not too large, then

$$\boldsymbol{H}(\bar{\boldsymbol{h}}+\Delta\boldsymbol{h}|_{\boldsymbol{k}'})\approx\boldsymbol{H}(\bar{\boldsymbol{h}}|_{\boldsymbol{k}'})+\left(\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{h}}\right)\bigg|_{\bar{\boldsymbol{h}}}\Delta\boldsymbol{h} \tag{25}$$

Therefore,

$$\Delta\boldsymbol{H}=\left(\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{h}}\right)\bigg|_{\bar{\boldsymbol{h}}}\Delta\boldsymbol{h}+\Delta\boldsymbol{H}_{0} \tag{26}$$

It can be seen from Eq. (26) that the error of the measured value results from both the uncertainty of the hydraulic head at the boundary and the uncertainty of the observed hydraulic head, and this is a difference from the conventional Bayesian method. If the correlation between the hydraulic head at the boundary and measurement error is ignored, the covariance of $\Delta\boldsymbol{H}$ can be derived from the following formula:

$$\boldsymbol{C}_{\Delta\boldsymbol{H}}=\left(\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{h}}\right)\boldsymbol{C}_{\boldsymbol{h}}\left(\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{h}}\right)^{\mathrm{T}}+\boldsymbol{C}_{\Delta\boldsymbol{H}_{0}} \tag{27}$$

where $\boldsymbol{C}_{\Delta\boldsymbol{H}}$ is the covariance of the measured hydraulic head and $\boldsymbol{C}_{\boldsymbol{h}}$ is the covariance of the hydraulic head at the boundary. Then, the covariance of the measurement error of the measured hydraulic head becomes

$$\boldsymbol{C}_{\Delta\boldsymbol{H}_{0}}=\boldsymbol{C}_{\Delta\boldsymbol{H}}-\left(\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{h}}\right)\boldsymbol{C}_{\boldsymbol{h}}\left(\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{h}}\right)^{\mathrm{T}} \tag{28}$$

In Eq. (28), $\dfrac{\partial\boldsymbol{H}}{\partial\boldsymbol{h}}$ can be derived from the stochastic finite element method.

As $\boldsymbol{C}_{\boldsymbol{H}^*}$ in the formulas of the Bayesian method is only the covariance of the error of the measured value, $\boldsymbol{C}_{\boldsymbol{H}^*}$ in Eqs. (14), (15) and (22) needs to be replaced with $\boldsymbol{C}_{\Delta\boldsymbol{H}_0}$ as expressed in Eq. (28) of the generalized Bayesian method.

4 Case study

A sluice foundation shown in Figure 1 is examined here. The infiltration domain is 42.00 m in depth and 74.00 m in width. The finite element mesh consists of 104 elements and 126 nodes. It was assumed that the media was homogeneous, the *x* and *y* directions were the main infiltration directions, and the true values of the permeability coefficients in the *x* and *y* directions were, respectively, $k_x = 3.0\times10^{-4}$ m/d and $k_y = 1.0\times10^{-4}$ m/d. The seepage in the foundation was regarded as a confined seepage field. According to the finite element analysis of a steady seepage field, the imaginary measured data of hydraulic head at different nodes were derived and are listed in the mean value column of Table 1. The statistics of measured data of the hydraulic head, the prior information of the permeability coefficient, and the hydraulic head at the boundary were also assumed and are listed in Tables 1 through 3.

Table 1 Statistics of measured data of hydraulic head

Number of nodes	Mean value (m)	Standard deviation(m)	Coefficient of variation
105	15.76	1.23	0.078
106	14.24	1.03	0.072
107	13.23	0.99	0.075
119	16.31	1.32	0.081

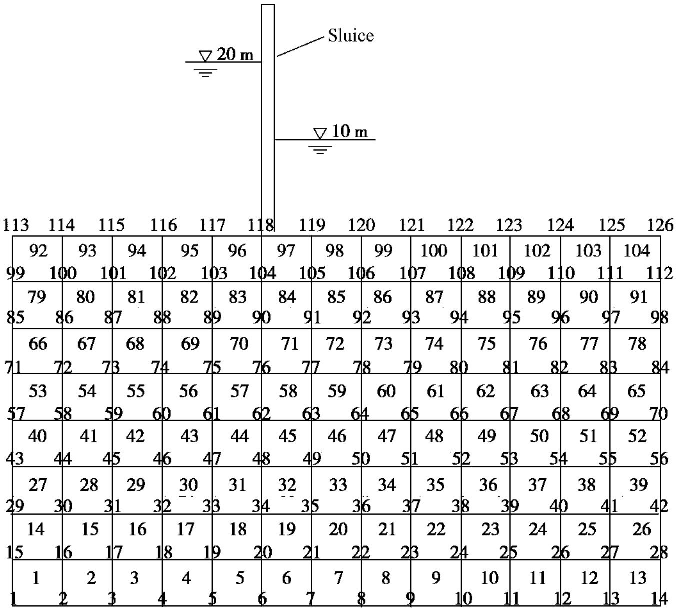

Figure 1 Finite element mesh of sluice foundation

Table 2 Statistics of prior information of permeability coefficient

Permeability coefficient	Mean value(10^{-4}m/d)	Standard deviation(10^{-4}m/d)	Coefficient of variation
k_x	2.90	0.87	0.30
k_y	0.90	0.27	0.30

Table 3 Statistics of hydraulic head at boundary of seepage field

Section	Mean value (m)	Standard deviation (m)	Coefficient of variation
Upstream	20.0	2.0	0.10
Downstream	10.0	1.0	0.10

A program was developed for the generalized Bayesian method based on SFEM and applied to stochastic back analysis of the permeability coefficient. The mean values of R, which is the ratio of k_x to k_y, determined by back analysis using the generalized Bayesian method and conventional Bayesian method, are listed in Table 4, and we can see that they basically coincide with the known true value ($R = k_x / k_y = 3$). The stochastic inversion results of the standard deviation of the permeability coefficient with different methods are given in Table 5.

It can be seen from Table 5 that the standard deviation and coefficient of variation derived from the conventional Bayesian method are small as compared with those derived from the Gauss-Markov method, in which the uncertainty of the permeability coefficient was not considered. However, when the uncertainty of the hydraulic head at the boundary was further considered in the generalized Bayesian method, the standard deviation and coefficient of variation became even smaller. Therefore, the generalized Bayesian method is superior to the other two methods, and it is most suitable for the stochastic back analysis of a seepage field.

Table 4 Mean of ratio of k_x to k_y determined by back analysis using generalized Bayesian method and conventional Bayesian method

Number	Iteration number	Iterative initial value of R	Generalized Bayesian method		Conventional Bayesian method	
			Inversion value of R	Relative error of R (%)	Inversion value of R	Relative error of R (%)
1	13	1.00	3.01	0.33	3.01	0.33
2	14	6.00	3.01	0.33	3.04	1.33

Table 5 Variability of permeability coefficient determined by stochastic back analysis using different methods

Permeability coefficient	Initial value of permeability coefficient (10^{-4}m/d)	Generalized Bayesian method		Conventional Bayesian method		Gauss-Markov method	
		Standard deviation (10^{-4}m/d)	Coefficient of variation	Standard deviation (10^{-4}m/d)	Coefficient of variation	Standard deviation (10^{-4}m/d)	Coefficient of variation
k_x	2.5	0.349	0.116	0.585	0.194	0.870	0.290
k_y	0.5	0.108	0.108	0.188	0.188	0.271	0.271

5 Conclusions

This paper has described the generalized Bayesian method for stochastic back analysis of the permeability coefficient. An example was presented to illustrate the method. The results indicate that the proposed method can account for the uncertainties of the measured hydraulic head, permeability coefficient and hydraulic head at the boundary. Both the mean and the standard deviation of the permeability coefficient can be obtained using the proposed method. The mean's accuracy is supported by comparison with the true value, and the standard deviation is less than that obtained using the conventional Bayesian method, so the generalized Bayesian method is valid and applicable. However, the present study is only a preliminary attempt to conduct stochastic back analysis of the permeability coefficient using the generalized Bayesian method. The following issues need further study:

(1) It is assumed that the permeability coefficient in the generalized Bayesian method follows a normal distribution within a range from $-\infty$ to $+\infty$. In engineering practice, the permeability coefficient has only positive values. Therefore, a lognormal distribution or the truncated normal distribution should be adopted for the stochastic back analysis of the permeability coefficient.

(2) This study conducted a stochastic back analysis of the permeability coefficient only for steady seepage. The back analysis of the permeability coefficient for unconfined and

unsteady seepage needs to be further studied.

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