Water Science and Engineering, 2009, 2(3): 78-86
doi:10.3882/j.issn.1674-2370.2009.03.008

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Reliability analysis method for slope stability based on sample weight

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Abstract: The single safety factor criteria for slope stability evaluation, derived from the rigid limit equilibrium method or finite element method (FEM), may not include some important information, especially for steep slopes with complex geological conditions. This paper presents a new reliability method that uses sample weight analysis. Based on the distribution characteristics of random variables, the minimal sample size of every random variable is extracted according to a small sample *t*-distribution under a certain expected value, and the weight coefficient of each extracted sample is considered to be its contribution to the random variables. Then, the weight coefficients of the random sample combinations are determined using the Bayes formula, and different sample combinations are taken as the input for slope stability analysis. According to one-to-one mapping between the input sample combination and the output safety coefficient, the reliability index of slope stability can be obtained with the multiplication principle. Slope stability analysis of the left bank of the Baihetan Project is used as an example, and the analysis results show that the present method is reasonable and practicable for the reliability analysis of steep slopes with complex geological conditions.

Key words: *reliability analysis; slope stability; sample weight coefficient; t-distribution; Bayes formula*

1 Introduction

China is now undergoing a period of large-scale construction. More and more steep slopes are appearing in a series of large-scale hydropower and traffic engineering projects, and the slope stability directly affects the reliability of these projects. It is therefore important to evaluate the safety degree of the slope stability. At present, the safety factor obtained according to the *Design Specification for Slope of Hydropower and Water Conservancy Project* (hereafter referred to as the *Specification*) (PSCG 2006), cannot reflect the actual significance of the project. For example, if the safety factor of a project equals 1.5, it does not mean that there is a 150% safety margin or 1.5 times the emergency capacity; it is related to the assumptions in the design, the method for determining the safety factor, and so on. In other

This work was supported by the National Natural Science Foundation of China (Grant No. 90510017).
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Received May 4, 2009; accepted Jul. 13, 2009

words, the value of the adopted safety factor varies with different concrete problems; even when the engineering design is the same, it varies with different slide blocks. The safety factor, as an index of the safety degree in numerical form, is actually the average or equivalent skid-resistance against the sliding force. The value of the safety factor obtained with the single safety factor method cannot be used as a quantitative scale to reflect the safety degree, because it is difficult to take all the factors related to the slope stability into consideration and the factor does not meet engineering requirements (Chen et al. 2001; Wang 1998; Matsui and San 1992; Griffiths and Lane 1999). Hence, it is more reasonable to use the reliability calculation method, in which a reliability index is adopted, to judge whether the slope is stable or not.

According to the *Specification* (PSCG 2006), the reliability must be evaluated based on the safety factor. It is assumed that uncertain parameters have a normal distribution. The 3σ criterion is used to estimate the standard deviation of uncertain parameters, which means that the coverage of an uncertain parameter, such as the friction coefficient *f*, ranging from $\mu_{\mathrm{f}} - 3\sigma_{\mathrm{f}}$ to $\mu_{\mathrm{f}} + 3\sigma_{\mathrm{f}}$, is 99.73% ($\mu_{\mathrm{f}}$ and σ_{f} are the average and standard deviation of the friction coefficient, respectively). In other words, the probability of the parameter meeting the most unfavorable condition is 99.73%. In general, the standard deviation of the parameter *f* is $\sigma_{\mathrm{f}} = \dfrac{f_{\mathrm{cb}} - f_{\mathrm{lb}}}{6}$, and the average is $\mu_{\mathrm{f}} = f_{\mathrm{cb}} - 3\sigma_{\mathrm{f}} = f_{\mathrm{lb}} + 3\sigma_{\mathrm{f}}$, where f_{cb} and f_{lb} are the upper and lower limit values of the uncertain parameter, respectively, and are determined based on engineering experience.

In the same way, based on the 1σ criterion with an expected value of 95%, we can also determine the two most unfavorable samples for each random variable to calculate the safety factor. It is assumed that there are *n* independent random variables in the normal distribution, and the statistical analysis is made of safety factor values based on the 2*n* samples. By calculating the characteristic values, the reliability index of slope stability can be obtained: $\beta = \dfrac{\mu_{\mathrm{F}} - 1}{\sigma_{\mathrm{F}}}$, where μ_{F} and σ_{F} are the average and standard deviation of the safety factor, respectively.

From the above analysis, we can obtain the following conclusions:

(1) The standard deviation is simply assumed without conforming to the statistical definition and without considering its effect on sample values under different possible conditions, which may lead to more errors and reduce the credibility of the calculated result.

(2) The sample size is small. The two samples for each random variable are insufficient for the reliability analysis.

(3) Although at a certain expected value of 95%, the approach of artificially setting the maximal range of random variables can meet all kinds of sample combination conditions, including the most unfavorable condition, it obviously has a large safety margin and also violates the engineering concept and reliability theory (Dawson et al. 1999; Li et al. 2003a; Zhang and Liu 2003; Jia and He 2003). According to the *Specification* (PSCG 2006), we

should first choose an expected value to reflect the most unfavorable condition. Thus, almost all the values of random variables meet the expected value. However, reliability analysis provides all the possible values of the random variables, and then analyzes the expected possibility that can meet the requirements.

Since there are some problems in the *Specification* (PSCG 2006), this paper presents a new reliability method based on sample weight analysis and tries to solve related problems with total probability weight.

2 Practical reliability analysis method and steps

2.1 Samples of random variables and weight coefficient of sample combination

The factors that influence the safety factor, such as rock mass deadweight, underground water pressure, and rock mass shear strength parameters, may be regarded as the random variables in the normal distribution. The sample weight coefficient of random variables can be defined as the frequency of occurrence or probability of a sample value.

It is assumed that there are *m* random variables and each random variable has *n* samples. For convenience, it is assumed that all random variables are independent. The weight of the *i*th sample of the *j*th random variable is $r_{i,j}\left(i=1,2,\cdots,n;\ j=1,2,\cdots,m\right)$, where $0 \le r_{i,j} \le 1$. If the distribution characteristics of a random variable are known and *d* samples of the random variable are randomly selected, then weight coefficients for each sample can be obtained according to the fractile of the random sample in the sample distribution in the following way:

First, a series of sample values of the random variable, $x_1, x_2, \cdots, x_d$, can be arranged from smallest to largest, where $x_1 \le x_2 \le \cdots \le x_d$. Then, we can divide the value range of the random variable into the following intervals: $\left(-\infty, x_1\right], \left(x_1, x_2\right], \left(x_2, x_3\right], \cdots, \left(x_{d-1}, x_d\right]$. According to sample distribution characteristics, the probability of occurrence of the random variable in each region, p_i, is calculated using the Bayes formula:

$$\begin{aligned} p_1 &= \int_{-\infty}^{x_1} f(x)\mathrm{d}x \\ p_i &= \int_{x_{i-1}}^{x_i} f(x)\mathrm{d}x \qquad (i=2,3,\cdots,d) \end{aligned} \tag{1}$$

where $f(x)$ is the density function of the random variable distribution. Weight coefficients corresponding to each probability are represented as follows:

$$r_i = \frac{p_i}{\sum\limits_{i=1}^{d} p_i} \qquad (i=1,2,\cdots,d) \tag{2}$$

If a sample distribution characteristic is unknown, its value can be selected according to engineering experience. Based on the total probability theorem and the Bayes formula for post-experience probability, the weight coefficients of samples, $r_{i,j}$, and sample combinations, $R_{i,j}$, have the following relationship:

$$R_{i,j}=\frac{R_{i,j-1}r_{i,j}}{\sum_{i=1}^{n}R_{i,j-1}r_{i,j}}\qquad (i=1,2,\cdots,d;\ j=2,3,\cdots,m) \tag{3}$$

where $R_{1,j}, R_{2,j}, \cdots, R_{d,j}$ are the weight coefficients of the sample combinations of *j* random variables, $R_{i,1}=r_{i,1}$, $\sum_{i=1}^{d}r_{i,j}=1$, and $\sum_{i=1}^{d}R_{i,j}=1$ $(i=1,2,\cdots,d;\ j=1,2,\cdots,m)$.

2.2 Determination of smallest sample size

According to the reliability concept in the *Specification* (PSCG 2006), the expected value of a certain random variable is given in advance. The optimal samples of the variable selected at the given expected value can meet different random conditions (Duan et al. 2002; Ayyub and Haldar 1984; Haldar and Reddy 1992; Martinez-Flores et al. 2008). In regards to the test sample size and the FEM calculation sample size, there are many factors, but it is impossible to take all of them into consideration. Generally speaking, with the goal of satisfying the requirement of engineering accuracy, it is hoped that the factors can be reduced to a minimum for the optimum design.

If the sample size in a sample test is small, the samples do not have normal distribution but can be considered to have a *t*-distribution. It is more reasonable to use the *t*-distribution than the normal distribution, because the normal distribution is a special case of the *t*-distribution when the sample number tends to infinity. At the confidence level of $1-\alpha$ of the sample size of a project (α is the probability of failure), the probability of the average of a random variable $\overline{X}$ falling into the interval $\left[\overline{X}-t_{\frac{\alpha}{2}}(n-1)\frac{\sigma}{\sqrt{n}},\ \overline{X}+t_{\frac{\alpha}{2}}(n-1)\frac{\sigma}{\sqrt{n}}\right]$ is $1-\alpha$. We can determine the minimal sample size at the confidence level of $1-\alpha$ based on the characteristics of the sample distribution.

With the goal of ensuring a certain level of accuracy, we need to extract a few samples from a larger number of samples to carry on numerical analysis (such as the FEM), so as to reduce the computational complexity. This can be regarded as the estimation of two normal general parameters, the object sample size and extracted sample size. We construct a statistic variable *T*:

$$T=\frac{\overline{Z}-\overline{Y}}{\left(ks_1^2+ls_2^2\right)^{\frac{1}{2}}}\left[\frac{kl(k+l-2)}{k+l}\right]^{\frac{1}{2}}\geq t_{1-\frac{\alpha}{2}}(k+l-2) \tag{4}$$

where $\overline{Y}$, s_2, and l are the average, variance, and sample number corresponding to the larger sample size, respectively; $\overline{Z}$, s_1, and k are the average, variance, and sample number corresponding to the extracted sample size, respectively; *k* and *l* are integers; and $k\leq l$. Then, we can obtain the minimal sample size *d* from the *k* value.

2.3 Statistical characteristics of safety factor and determination of reliability index

According to Eq. (4), we extract the minimal sample size of random variables to make random combinations, which are used as the input values for the FEM analysis. We calculate weight coefficients $R_i\,(i=1,2,\cdots,n')$ for each random sample combination with Eq. (3), where n' is the total number of the sample combination. Using the finite element strength reduction (Li et al. 2003b), the values of output parameters corresponding to each random sample combination, such as the safety coefficients $F_i\ (i=1,2,\cdots,n')$ for anti-slide stability analysis, are acquired. According to the total probability formula, the average and the variance of the general safety factor F can be obtained as follows using the multiplication formulas:

$$\mu_{\mathrm{F}} \approx \overline{F} = \sum_{i=1}^{n'} R_i F_i \tag{5}$$

$$\sigma_{\mathrm{F}} = \sqrt{\frac{1}{n'}\sum_{i=1}^{n'}\left(F_i - \overline{F}\right)^2} \tag{6}$$

We can obtain the reliability index as follows:

$$\beta = \frac{\mu_{\mathrm{F}} - 1}{\sigma_{\mathrm{F}}} \tag{7}$$

2.4 Steps of practical reliability analysis method

Random variables, such as the rock density, rock shear strength parameters, friction coefficient, and cohesive force, may be regarded as the important factors for the reliability index. Based on the numerical characteristic of random variables and the small sample *t*-distribution estimation, we can find the minimum sample size at a given expected value. Using different sample combinations as the input, we can obtain a series of output values. Using the conditional probability and the Bayes formula, the weight coefficients corresponding to each sample combination can be obtained. Based on the total probability theorem, the reliability index can be obtained by multiplying the output values and the weight coefficients. We call this method the practical reliability analysis method. The steps are as follows:

(1) The number of random variables is determined: The uncertain factors affecting the stability index are considered the random variables, and certain weights are assigned to each variable according to engineering experience. With the goal of meeting the engineering requirements, we find the minimal number of random variables by solving Eq. (4) under the expected value $1-\alpha$ of the *t*-distribution.

(2) The minimal sample size of each random variable is determined: Based on the engineering requirements, we find the minimal sample size of each random variable by solving Eq. (4) under the condition that the *t*-distribution has an expected value $1-\alpha$.

(3) Weight coefficients, $r_{1,j}, r_{2,j}, \cdots, r_{n,j}$, corresponding to each sample of each random

variable, $x_{1,j}, x_{2,j}, \cdots, x_{n,j}$, are obtained according to engineering experience or the distribution characteristics of random variables.

(4) The sample values of each random variable at minimal sample size are randomly selected and combined. The weight coefficient corresponding to each sample combination is determined with Eq. (3).

(5) Each group of sample combinations is used as a system input for the FEM analysis of the slope stability. The output safety coefficients F_i corresponding to each sample combination can be obtained.

(6) According to the random distribution characteristic of the general safety factor F, the reliability index for the slope stability analysis is determined with Eqs. (5) through (7).

3 Engineering example

There is a steep slope with complex geological conditions on the left bank of the Baihetan Hydraulic Project, with a rock density of 2 850 kg/m^3, an elasticity modulus of 2.2×10^6 kPa, and a Poisson ratio of 0.22. The three-dimensional finite element meshes of a slide block are shown in Fig. 1. We conducted the reliability analysis of the stability of this slide block by conceptualizing it as being composed of fault e_1, fault e_2, and fracture J_1. The bedding fault zone C_1 is the bottom slip surface.

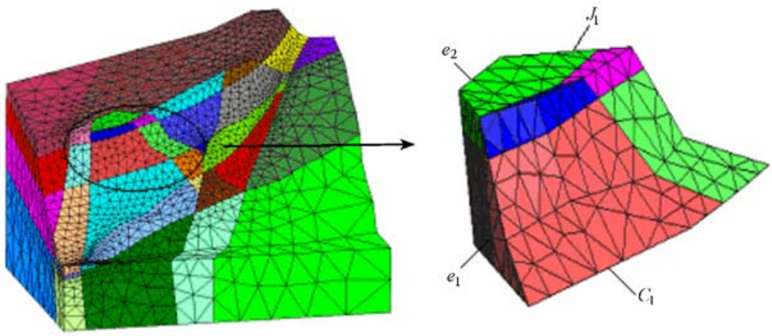

Fig. 1 3-D finite element meshes of slide block

Random variables that influenced the stability and safety of the slide block were analyzed. We assumed there were eight random variables: the friction coefficient f_i (i = 1, 2, 3, 4) and the cohesive force c_i (i = 1, 2, 3, 4) for structural planes e_1, e_2, J_1, and C_1, respectively. The numerical characteristics of the random variables were obtained through experiments. We assigned each of them an initial weight according to engineering experience and conducted sensitivity analysis for each variable with the single safety factor method. From the small sample t-distribution of the random variables at a given expected value of 95%, we found that the minimal number of random variables to be considered in the calculation is two. We selected two random variables, the friction coefficient f_4 and the cohesive force c_4 of the bottom slip surface C_1.

Based on the test samples, we calculated the numerical characteristics of the friction coefficient f_4 and cohesive force c_4. The values were $\mu_{f_4} = 0.35$, $\mu_{c_4} = 0.05$ MPa, $\sigma_{f_4} = 0.021$, and $\sigma_{c_4} = 0.0015$ MPa. Assuming the two variables have normal distribution at a certain expected value (the expected values of both variables were set at 90%), the minimal sample size of each variable could be obtained: four for the friction coefficient f_4 and eight for the cohesive force c_4. The sample values of each variable were selected according to the numerical characteristics of the random variables, which should be discrete values. The weight coefficients of each sample of the two variables were obtained with Eqs. (1) and (2). The sample values and corresponding weight coefficients of f_4 and c_4 are shown in Table 1.

Table 1 Sample values and corresponding weight coefficients of two variables

Random variable	Serial number	Sample value	Weight coefficient	Serial number	Sample value	Weight coefficient
Cohesive force	1	0.048 8	0.091 3	5	0.052 4	0.215 3
	2	0.050 4	0.110 5	6	0.049 0	0.038 6
	3	0.048 0	0.220 0	7	0.051 3	0.105 0
	4	0.051 1	0.090 8	8	0.051 9	0.128 5
Friction coefficient	1	0.356 9	0.274 3	3	0.365 2	0.208 3
	2	0.346 1	0.204 0	4	0.337 6	0.313 4

Note: Units of the sample values of cohesive force are MPa.

By analyzing 32 sample combinations with the finite element strength reduction method, we obtained the safety coefficients F_i ($i = 1, 2, \cdots, 32$), corresponding to each sample combination. The weight coefficients of different sample combinations were obtained with Eq. (3). Here, the Coulomb-Mohr criterion was adopted as the constitutive relation in geotechnical engineering. If the calculated results fail to converge during the finite element calculation, we consider the slope unstable. The safety coefficients F_i and weight coefficient R_i corresponding to different sample combinations are shown in Table 2. Using Eqs. (5) through (7) we obtained the reliability index of the slide block: $\beta = \dfrac{\mu_{\mathrm{F}} - 1}{\sigma_{\mathrm{F}}} = \dfrac{1.129 - 1}{0.123} = 1.05$, with a relative failure possibility of 14.7%.

Table 2 Safety coefficients and weight coefficients for different sample combinations

Serial number	Weight coefficient	Safety coefficient	Serial number	Weight coefficient	Safety coefficient	Serial number	Weight coefficient	Safety coefficient
1	0.069	0.904	12	0.023	1.111	23	0.035	1.206
2	0.029	0.911	13	0.019	1.120	24	0.059	1.209
3	0.012	0.939	14	0.021	1.124	25	0.046	1.214
4	0.035	0.975	15	0.026	1.130	26	0.019	1.223
5	0.028	0.995	16	0.044	1.132	27	0.008	1.240
6	0.033	1.012	17	0.060	1.134	28	0.023	1.268
7	0.040	1.019	18	0.025	1.154	29	0.019	1.276
8	0.067	1.030	19	0.011	1.162	30	0.022	1.286
9	0.045	1.046	20	0.030	1.172	31	0.027	1.312
10	0.019	1.060	21	0.025	1.184	32	0.045	1.321
11	0.008	1.096	22	0.029	1.196			

4 Conclusions

Using the new practical reliability analysis method to evaluate the safety degree of slope stability based on sample weight analysis, we can draw the following conclusions:

(1) Practical reliability analysis is a method in which the semi-empirical weights are determined based on the total probability theorem. Using the Bayes formula, we can consider all factors during the selection of the minimal sample size, which improves the reliability of samples.

(2) Based on the small sample *t*-distribution, the minimal sample size at a certain expected value can be obtained. This involves selection of the number of variables to be considered and the minimal sample size for each variable.

(3) With the FEM and numerical statistics based on the weight theorem of samples and their combination, the reliability index can be obtained using the Bayes formula and the total probability formula, significantly reducing the calculation workload.

(4) With the total probability idea, the semi-empirical weights-based practical reliability analysis method can meet the demands of different precisions (for example, different expected values). From this point of view, this method is more feasible and reasonable than methods based on the single safety factor criterion.

The numerical example of the engineering project proved the correctness and feasibility of the practical reliability analysis method.

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